SELF-ORGANIZATION AND NONEQUILIBRIUM STRUCTURES IN THE PHASE SPACE

Article *in* International Journal of Modern Physics B · January 2012

DOI: 10.1142/S021797920803937X

Sergei Kruchinin
National Academy of Sciences of Ukraine

International Journal of Modern Physics B
Vol. 22, No. 13 (2008) 2025–2045

World Scientific
www.worldscientific.com

SELF-ORGANIZATION AND NONEQUILIBRIUM STRUCTURES IN THE PHASE SPACE

V. NOVIKOV*, S. KRUCHININ†, N. N. BOGOLUBOV JR.‡ and S. ADAMENKO§

*Institute for Electrophysics & Radiation Technology of NASU, Kharkiv, Ukraine
†Bogolyubov Institute for Theoretical Physics, Kyiv, Ukraine
‡V.A. Steklov Mathematical Institute, Moscow, Russia
§Electrodynamic Laboratory "Proton-21", Kyiv, Ukraine
†skruchin@i.com.ua

Received 27 November 2007

We consider the possibilities of the formation of quasistationary distributions of particles over energy with power asymptotics in nonequilibrium systems and dynamical systems with couplings. It is shown that the Tsallis distribution is related to the exact solutions of a kinetic equation of the Boltzmann type and those of covariant kinetic equations of the Vlasov nonlocal statistical mechanics.

We have studied the connection of the power-like solutions of kinetic equations with the eigenfunctions of fractional integro-differential operators and Jackson operators within quantum analysis, and that of nonextensiveness parameters in the framework of the Tsallis thermostatics, with flows in the phase space. It is shown that the processes running in a nonequilibrium nonconservative medium can be described by the solutions of the equation with fractional derivatives or Jackson derivatives for oscillations.

Keywords: Nonlinearity; nonequilibrium systems; nonlocal statistical mechanics; kinetic equations.

1. Introduction

As an important achievement of physics for the last decade, we mention the development of the theory of self-organization and synergetics, which render a permanent influence on the studies in many branches of science. Moreover, the appearance of structures as a result of the evolution of complex systems is the object of constant attention. One of the main tools of the analysis of structures is nonequilibrium thermodynamics.[1]

The achievements of the modern pulse technique in the derivation of great densities of energy and power in pulses of different nature, allowing one to use the concentrated flows of energy for the initiation of the processes of self-organization in systems of many particles, and for the creation of structures in a very wide range of spatial and energy scales. The least studied and, probably, most interesting field for the analysis of mechanisms of the origin and the evolution of structures is the research of the processes of self-organization in nuclear physics. The urgency of studies in this field and the necessity of new models in the theoretical investigation of nuclear structures become clear from the analysis of the long-term European program of nuclear studies.[2]

Although the theoretical studies demonstrating the necessity of the analysis of fractal structures in the nuclear matter (Refs. 3–5) and the results of experimental studies of the appearance of nuclear structures[6–8] are constantly published, the necessity of the general theoretical approach to the self-organization and the thermodynamics of locally nonequilibrium systems is only strengthened.

From our viewpoint, more and more realizable now is the construction of a general theoretical conception allowing the analysis of the origin and the evolution of structures under conditions of the essential nonlocality and irreversibility of the multiparticle interaction in the framework of the kinetics and the nonequilibrium thermodynamics of complex systems with couplings. In the theory of self-organizing nuclear processes, one of the most important problems is the definition of the principles of the evolution (variational principles) allowing one to determine the energy trend of the evolution (i.e., the sign of the arising integral mass defect) of macroscopic nuclear systems under significantly nonequilibrium conditions. The complexity consists in that the systems of particles, which are of interest to us, are nonideal systems with nonholonomic couplings. Hence, neither the variational principles of the Hamilton dynamics nor the ordinary thermodynamic principles of the evolution are, strictly saying, suitable.

The principles of the Prigogine nonequilibrium thermodynamics, which are traditionally used in the theory of self-organization, are based on the wide application of the notion of locally equilibrium states to the description of thermodynamical systems and inadequately describing the physical processes running in the systems undergoing the action of a mass force, i.e., the action which is quasihomogeneous inside the region occupied by particles, rather than an action concentrated only on its surface. According to the Prigogine approach, the evolution occurs over locally equilibrium states, and the notion of "strongly nonequilibrium state" means a state in which the values of macroscopic parameters of a system (such as the density, temperature, etc.) in locally equilibrium states are strongly inhomogeneous in space. Such an approach leads to the decisive (for the evolution of a system) role of flows in the coordinate space, which are related to the gradients of macroscopic parameters.

In modern experimental setups, such levels of intensities of the flows of particles and energy in systems of particles are available,[9] at which the evolution occurs over locally nonequilibrium states of the system. Moreover, the defining physical quantity for all the elements of the system is the flow in the phase space of the system, rather than a flow in the coordinate space.

In similar situations, the equilibrium states are absent, even in physically infinite small regions of space. The nonlocal collective states of particles appear, the properties of similarity of the system on all scales become significant, and the strong correlations in the phase space are manifested.

The key role of correlations in the systems of many particles was first clearly shown in the fundamental work,[10] and Vlasov was among the first to indicate the necessity of the account of nonlocal properties of the dynamics observed in the systems of many particles.[11] He introduced the notion of collective variables into physics and constructed a nonlocal statistical theory. The absence of locality leads to the face that the influence on the physical processes is not limited by an infinitely small neighborhood of the point under consideration, but requires us to account for the finiteness of the interaction region and, hence, to consider all the orders of the differentials of variables describing a state of the system. There exist two approaches to the description of these effects.

In the first case, the functional and integro-differential equations with regard to delay are used for the description of the evolution of the system, rather than differential equations. This approach was developed by Volterra.[12]

In the other case, a specific space of support elements is used to describe the nonlocality, rather than the 4-dimensional space-time.[11] In Vlasov's opinion, the statistical theory of dynamical systems should be constructed in the Cartan space,[13] which is the totality of points in the 4-dimensional Riemann space and the tangent planes of different orders at every point. That is, the basic variable of statistical theory are not only the coordinates, time, and velocities (momenta), but also the accelerations of all orders.

The studies of many researchers in the field of kinetics and nonequilibrium thermodynamics allow us to conclude that one of the very essential results of the manifestation of nonlocality and nonholonomic couplings in dynamical systems is the appearance of self-similar states in the phase space of a dynamical system of particles. This conclusion follows from the fact that the power and power-like distribution functions of particles over energy arise in a wide diversity of approaches to the description of nonequilibrium systems.

As shown in Ref. 11, the nonlocality of statistical mechanics yields the Vlasov covariant kinetic equation with self-consistent fields in the Cartan space, whose exact solutions possess the power asymptotics.[11,14] It was first shown in Refs. 15–17 that the Boltzmann kinetic equations in a homogeneous space can also possess the exact power of locally nonequilibrium solutions, corresponding to steady flows in the phase space. Currently, the thermodynamic properties of a system of many particles, being in quasistationary nonequilibrium states with great correlations, are often studied in the framework of the formalism of nonextensive thermostatics[18] which is based on the application of the Tsallis quasipower distribution function and the nonextensive generalization of entropy.

The substantiation of the use of such a nonextensive distribution function is the important problem, and, as usual, its solution is based exclusively on the thermodynamical arguments. The systems of many particles, which are characterized by the distribution functions over energy with power asymptotics, are defined by the peculiarities of nonholonomic couplings and flows in the phase space of the dynamical system. As a result of the existence of such couplings, the system of interacting particles turns out to be irreversible and, from the thermodynamical viewpoint, open.

In the present work, we will show that the distribution functions of a system of many interacting particles, which are similar to those used in the Tsallis nonextensive thermostatics, possess both the thermodynamic and kinetic foundations. We will analyze the properties of nonlocality (and nonideality) of the oscillatory processes in nonextensive states.

It will be shown that the systems with power-like distributions are nonconservative on the macroscopic level on the fulfillment of the energy conservation law on the level of binary interactions. The nonconservative properties of such flow systems are well-described with the use of generalized derivatives (fractional derivatives[19,20] and in the framework of quantum q-analysis.[21]). Such an approach to the description of the evolution of nonideal media with the use of fractional derivatives (or quantum analysis) allows one to formulate the evolution of dynamical systems on the basis of generalized variational principles (see, for example, Ref. 22).

2. Derivation of the Tsallis Nonextensive Distribution Function from the Microcanonical Distribution

In the framework of the thermodynamic approach, we will show that the power distribution functions arise on the account of the finiteness of the number of states of the system and the thermostat (see, for example, Ref. 23).

Indeed, let the system Σ_1 and the thermostat Σ_2 under study contain, respectively, N_1 and N_2 particles and be described by canonical variables x_1 and x_2. We assume that

$$N_2 \gg N_1 \,. \tag{1}$$

Let the collection of the canonical variables of the thermostat x_2 consist of the coordinates q_2 and all the momenta of the thermostat p_2. We present the thermostat energy in the form of a sum of the kinetic and potential energies $H(x_2) = K_2(p_2) + U_2(q_2)$.

We can consider the joint system (Σ_1 and the thermostat) to be isolated. Therefore, it can be described by the microcanonical distribution

$$w(x_1, x_2) = \frac{1}{\Omega(E)}\delta(E - H(x_1, x_2))\,, \tag{2}$$

where the Hamiltonian of the joint system consists of the Hamiltonians of both subsystems and the interaction energy $U_{12}(x_1, x_2)$:

$$H(x_1, x_2) = H_1(x_1) + H_2(x_2) + U_{12}(x_1, x_2)\,. \tag{3}$$

It is obvious that the phase probability density of the system under study is

$$w(x_1) = \int_{(x_2)} w(x_1, x_2) dx_2 \tag{4}$$

by the theorem of the addition of probabilities. We assume that there exists the limit $E/(N_1 + N_2) = (3/2)T$, and the inequalities $U_{12} \ll H_1, H_2$ and $H_1(x_1) \ll E$ are valid.

Then, after the direct calculation of integral (4), we get

$$w(x_1) \approx \frac{1}{\Omega(E)} \int_{(x_2)} \delta(E - H_1(x_1) - K_2(p_2) - U_2(q_2)) dq_2 dp_2 \,. \tag{5}$$

By integrating with respect to the momentum variables dp_2 of the thermostat, we can represent the function $w(x_1)$ as

$$\begin{aligned} w(x_1) &= \frac{1}{\Omega(E)} \int_{(q_2)} \Omega_k(E - H_1(x_1) - U_2(q_2)) dq_2 \,, \\ \Omega_k(E) &= \frac{1}{\Omega(E)} \int_{(p_2)} \delta(E - K_2(p_2)) dp_2 \,. \end{aligned} \tag{6}$$

The quantity $\Omega_k(E)$ is the derivative of the volume $\Gamma_k(E)$ contained in the hypersurface of a given kinetic energy:

$$\Omega_k(E) = \frac{d\Gamma_k(E)}{dE} \,, \quad \Gamma_k(E) = \int_{(K_2(p_2) \le E)} dp_2 \,. \tag{7}$$

By calculating the multiple integrals in Eq. (7), we obtain the relations $\Omega_k(E) = bE^{\frac{3N_2}{2}-1}$ and $b = (3N_2/2)a$. With their help, we transform Eqs. (6) to the form

$$w(x_1) = \frac{b}{\Omega(E)} \int_{(q_2)} (E - H_1(x_1) - U_2(q_2))^{\frac{3N_2}{2}-1} dq_2 \,. \tag{8}$$

Introducing the notations $(1/1 - q) = (3N_2/2) - 1$, $E = (1/1 - q)T$, and $B(E, q) = (bE^{\frac{1}{1-q}}/\Omega(E))$, we can represent the distribution function as

$$\begin{aligned} w(x_1) &= D(T, q) \exp_q \left(-\frac{H_1(x_1)}{T} \right), \\ D(T, q) &= B(T, q) \int_{(q_2)} \left(1 - (q-1) \frac{U_2(q_2)}{T - (q-1)H_1(x_1)} \right)^{\frac{1}{q-1}} dq_2 \,. \end{aligned} \tag{9}$$

In this formula, we used the q-generalization, $\exp_q(x)$, of the exponential function. Its inverse function $\ln_q(x)$ appears as

$$\exp_q(x) = (1 + (1-q)x)^{\frac{1}{(1-q)}} \,, \quad \ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \,. \tag{10}$$

The functions $\exp_q(x)$ and $\ln_q(x)$ are transformed, respectively, into the ordinary exponential function and the ordinary logarithm as the parameter q tends to unity.

Thus, for a system of particles, the quasistationary distribution, which appears on their interaction with the thermostat including a finite number of particles, turns out to be quasipower and corresponds to nonextensive states with great correlations.

In what follows, we will study the physical circumstances which lead to quasistationary states with similar properties on the basis of the exact solutions of kinetic equations, rather than on the basis of the thermodynamic relations.

3. Power Stationary Distribution Functions as the Exact Solutions of Kinetic Equations

In the works of Refs. 15–17, the power distributions were obtained as the exact solutions of the spatially homogeneous Boltzmann equation

$$\frac{\partial f(\mathbf{v})}{\partial t} = I_{st}\{f,f\} + \Psi(v)\,, \quad I_{st}\{f,f\} = \int d\mathbf{v}_1 d\Omega g\sigma(g,\theta)(f'f_1' - ff_1)\,. \tag{11}$$

Here, the standard notations $f = f(\mathbf{v},t)$, $f_1 = f(\mathbf{v}_1,t)$, $f' = f(\mathbf{v}',t)$, $f_1' = f(\mathbf{v}_1',t)$ for a binary collision and a scattering of particles with velocities $(\mathbf{v},\mathbf{v}_1) \leftrightarrow (\mathbf{v}',\mathbf{v}_1')$ are used. In Eq. (11), $g = |\mathbf{v} - \mathbf{v}_1|$ — the relative velocity which turns on the collision by an angle θ, $\sigma(g,\theta)$ — the scattering cross-section depending on the form of the intermolecular interaction, $d\Omega$ — an element of the solid angle, $I_{st}\{f,f\}$ — the operator describing the collision of particles, and $\Psi(v)$ — the term representing sources and sinks in the velocity space.

The exact stationary solutions of Eq. (11) arise in the presence of steady nonzero flows of particles (or energy) in the phase space. These flows are also related to the availability of sources and sinks of energy (or particles) which are uniformly distributed in the coordinate space and are concentrated in the phase space. The action of such sources and sinks on the system of particles is similar to the action of a mass force in the dynamical systems of mechanics. In this section, we will show two qualitatively different cases of the appearance of power distributions. First, we will demonstrate by the direct calculation that the Lenard–Balescu equation[24] has power solutions.

3.1. *Power solutions of the Lenard–Balescu equation*

The formation of quasistationary nonequilibrium distributions becomes most transparent on the use of the kinetic equations for an isotropic distribution function of particles in the form of a differential equation of continuity in the energy space. In this case, the kinetic equation with a collision integral in the Lenard–Balescu form (see, for example, Ref. 24) can be written in the form of the conservation law of the number of particles:

$$\frac{\partial f}{\partial t} + \mathrm{div}(\mathbf{j}) = \psi(p,f)\,, \quad I_{st} = -\mathrm{div}(\mathbf{j})\,, \quad j_i = D_{ij}\frac{\partial f}{\partial p_j} + F_i f\,. \tag{12}$$

The components of the flow vector of particles j_i in the phase space are defined in terms of the diffusion coefficient in the phase space D_{ij} and the friction force F_i. These quantities depend on the distribution function and the dispersion properties of the medium according to the relations

$$\begin{aligned} D_{ij} &= -2e^4 m \int d\mathbf{k}\frac{k_i k_j}{k^4}\,\frac{1}{|\varepsilon(\mathbf{kv})|^2}\int d\mathbf{p}' f(\mathbf{p}')\delta(\mathbf{k}(\mathbf{p}-\mathbf{p}'))\,, \\ F_i &= 2e^4 m \int d\mathbf{k}\frac{k_i k_j}{k^4}\,\frac{1}{|\varepsilon(\mathbf{kv})|^2}\int d\mathbf{p}'\frac{\partial f(\mathbf{p}')}{\partial p_j'}\delta(\mathbf{k}(\mathbf{p}-\mathbf{p}'))\,. \end{aligned} \tag{13}$$

Consider the sources and sinks localized in the neighborhood of certain energies and assume that they are absent in the remaining extensive region of energies. Then it is obvious that the stationary states are determined from the equation $\mathrm{div}(\mathbf{j}) = 0$ or, hence, from the equation $4\pi p^2 j_p = P$.

The zero values of a flow in the phase space P lead to a solution in the form of the equilibrium distribution functions which correspond to the detailed balancing principle. In the case where the flow has nonzero values, this equation also possesses the solutions with power asymptotics, which can be shown by the direct calculation of the flow for the power distribution functions.

Let a distribution function have a power form. For the distribution function with power asymptotics $f = Ap^S$, the dielectric permittivity of the medium can be calculated and has the form[15]

$$\varepsilon(\omega, k) = 1 - \frac{16\pi^2 m^2 A e^2}{\omega^2} \left|\frac{\omega m}{k}\right|^S \left(\frac{\omega}{k}\right)^3 \alpha(S) + i\frac{8\pi^3 m^2 A e^2}{\omega^2} \left|\frac{\omega m}{k}\right|^S \left(\frac{\omega}{k}\right)^3 ; \qquad \alpha(S) = \int_0^\infty \frac{d\xi \xi^{S+2}}{1-\xi^2} . \tag{14}$$

For the power asymptotics of the distribution function, the direct calculation (with the use of relations (14)) gives the formulas for the "force" and "the diffusion coefficient" in the phase space as functions of the exponent s and the normalizing constant A of the distribution function:

$$F_i = -\frac{4\pi m A e^4}{Bp^2} \int \frac{d\mathbf{k}}{k^3} k^4 \left|\frac{\mathbf{kp}}{k}\right|^s \frac{1}{\left|\frac{\mathbf{kp}}{k}\right|^{2s+2}} \left(\frac{\mathbf{kp}}{k}\right)^2 p_i = -\frac{4\pi^2 m A e^4 k_{\max}^4}{B} p^{-s-2} p_i \frac{1}{1-s}$$

$$D_{ij} = -\frac{2\pi m A e^4}{(S+2)p^2 B} \int \frac{d\mathbf{k}}{k^5} \left|\frac{\mathbf{kp}}{k}\right|^{S+2} \frac{k^4}{\left|\frac{\mathbf{kp}}{k}\right|^{2S+2}} \times \left\{\delta_{ij}[k^2p^2 - (\mathbf{kp})^2] + [3(\mathbf{kp})^2 - k^2p^2]\frac{p_i p_j}{p^2}\right\} . \tag{15}$$

Then the total flow in the phase space $J_p = 4\pi p^2 j_p$ takes the form

$$J_p = 4\pi p^2 \frac{\pi A e^4 (2m)^{1-s}}{\omega_p^4 m^4} k_{\max}^4 p^{2s+5} \left[p\frac{\partial f}{\partial p} \frac{1}{(s+1)(2s+9)} - \frac{2}{(2s+7)} f\right] = \frac{4\pi^2 A e^4 (2m)^{1-2s}}{2\omega_p^4 m^4} k_{\max}^4 p^{4s+7} \frac{(4s+9)}{(s+1)(2s+9)(2s+7)} . \tag{16}$$

The calculation of the divergence of this expression gives a collision integral in the form:

$$I_{st} = \frac{16\pi A^2 e^4 k_{\max}^4}{\omega_p^4 m} \varepsilon^{2(s+1)} \frac{(4s+7)(4s+9)}{(s+1)(2s+9)(2s+7)} . \tag{17}$$

Relation (21) yields the existence of two solutions: $s = -9/4$ and $s = -7/4$. As seen from Eq. (16), the solution $s = -7/4$ corresponds to the steadiness of the flow of particles in the phase space. It is easy to show that the second solution corresponds to the steadiness of the energy flow in the phase space. The calculated exponents are in agreement with the general relations obtained in the work[15] of the following form:

$$s = s_1 = -(\alpha_W + 3d)\frac{1}{2\beta}\,, \quad s = s_0 = s_1 + 1/2\,. \tag{18}$$

According to these relations, the exponents depend only on the degree of similarity of the matrix element of interaction α_W, the degree of similarity of the dependence of the energy on the momentum β, and the dimensionality of the momentum space d, and do not depend on other parameters, for example, on the number of degrees of freedom, like the case of the thermodynamic function obtained in Sec. 3.1.

However, the kinetic equations also have the solutions with somewhat different structure, which is characterized by a new type of the dependence of the exponent on the parameters of the problem. Namely, the exponent is determined by the magnitude of a steady flow in the phase space. We will demonstrate the existence of such solutions for the Boltzmann kinetic equation for particles with the Maxwell-type interaction potential (Maxwell molecules, the polarization interaction, etc.).

3.2. *Power solutions of the kinetic equation for Maxwell molecules*

For the Maxwell-type interaction, we have $g\sigma(g,\theta) \approx \alpha(\cos\theta) \approx \text{const.}$ in Eq. (11). The collision integral for such interactions possesses a number of specific features (see Refs. 25 and 26). In particular, it is convenient to analyze the kinetic equation (11), by using integral transformations on the basis of the Fourier transformation:

$$\begin{aligned} \Phi(y) &= \hat{R}\{F(x)\} = \int_0^\infty dx\, {}_0F_1\left(\frac{3}{2}, -xy\right) F(x)\,, \\ F(x) &= \hat{R}^{-1}\{\Phi(y)\} = \frac{1}{\Gamma^2\left(\dfrac{3}{2}\right)} \int_0^\infty dy\, {}_0F_1\left(\frac{3}{2}, -xy\right) \sqrt{xy}\,\Phi(y)\,. \end{aligned} \tag{19}$$

Here, $\Gamma(\alpha)$ is the gamma-function, ${}_0F_1(3/2, -xy) = (1/2\sqrt{xy})\sin(2\sqrt{xy})$, ${}_pF_q$ is the generalized hypergeometric function).

As follows from the definition, $\Phi(y,t)$ is the generating function for the moments:

$$\begin{aligned} \Phi(y,t) &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} M_n(t) y^n\,, \quad M_n(t) = \frac{1}{\left(\dfrac{3}{2}\right)_n} \int_0^\infty F(x,t) x^n dx\,, \\ (\alpha)_n &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,. \end{aligned} \tag{20}$$

The stationary nonequilibrium states are determined by the equation which follows from Eq. (4) and has the form $\Pi\{F, F, x\} = \int_0^x dx'\Psi(x')$.

The integral transformation (19) reduces the kinetic equation (11) to an integral equation in the form of a convolution for the function $\Phi(y)$:

$$\frac{\partial\Phi}{\partial t} - \frac{1}{y}\int_0^y d\xi\Phi(\xi)\Phi(y-\xi) = -M_{0\,st}\Phi + \Psi_R(y)\,, \quad [\Psi_R(y) = \hat{R}\{\psi(x)\}\,. \tag{21}$$

Here, $M_{0\,st}$ is a stationary value of the zeroth moment of the distribution function. The Laplace transformation $\hat{L}_z\{\Phi(y)\} \equiv G(z) = \int_0^\infty dye^{-zy}\Phi(y)$ reduces Eq. (21) to a differential equation which has the form of the Riccati equation

$$M_{0\,st}\frac{\partial G}{\partial z} + G^2 + \psi(z) = 0\,, \quad \psi(z) = \hat{L}_z\{y\psi(y)\} \tag{22}$$

for stationary states. Since $\Phi(y,t)$ is the generating function for the moments of the distribution function, it is easy to obtain a closed chain of evolutionary equations for any higher normed moments, by substituting the expression for $\Phi(y,t)$ in terms of the moments of the distribution function (20) in the integral equation (21). All moments of order $k < n$ can be obtained for an arbitrarily great n.

Let us obtain the asymptotics of the distribution function. By applying the integral transformations (19) and the Laplace transformation for δ-like sources and sinks, we obtain the contributions to the equation for $\Phi(y)$ and $G(z)$:

$$\begin{aligned}\psi_i^\delta &= Q_i\,{}_0F_1\left(\frac{3}{2}; -x_i y\right) = \frac{Q_i}{2\sqrt{x_i y}}\,\sin(2\sqrt{x_i y})\,\psi_i^\delta(z)\\ &= \hat{L}_z(y\psi_i^\delta) = Q_i\,\frac{1}{z^2}\,{}_1F_1\left(2; \frac{3}{2}; -\frac{x_i}{z}\right).\end{aligned} \tag{23}$$

We will consider the case of a single wide inertial interval. In these cases, the flow functions are simplified and have the asymptotics $\psi(z) = P/z^2$ and $P = \pm Q_i$. Then the kinetic equation for $G(z)$ in the inertial interval takes the form of the Riccati equation $M_{0\,st}(\partial G/\partial z) + G^2 + (P/z^2) = 0$, whose solutions are determined by the relation

$$G(z) = \frac{M_0}{z}\,\frac{c_1 s_1 z^{s_1} + c_2 s_2 z^{s_2}}{c_1 z^{s_1} + c_2 z^{s_2}}\,; \quad s_1 = \frac{1-v}{2}\,; \quad s_2 = \frac{1+v}{2}\,; \quad v = \sqrt{1 - \frac{4P}{M_0^2}}\,. \tag{24}$$

Here, c_1 and c_2 are arbitrary constants. By expanding $G(z)$ in a generalized power series in z^{-v} and by performing the inverse Laplace transformation termwise, we get

$$\Phi(y) = cM_0 s_1 + M_0(s_2 - cs_1)E_v\left(-\frac{y^v}{c}; 1\right). \tag{25}$$

Here, $E_v(x;\mu) = \sum_{k=0}^\infty (x^k/\Gamma(\mu + k(1/v))$ is the Mittag–Leffler entire function.

In order to find the distribution function, it is necessary to carry out the inverse Laplace transformation. Preliminarily, we introduce the new function $\Phi_s(y) =$

$e^{sy}\Phi(y)$ which coincides with $\Phi(y)$ as $s \to 0$. Now, the inverse transformation of $\Phi_s(y)$ can be performed termwise with the expansion of $E_v(x;1)$ in a series. Thus, we obtain the distribution function as

$$F_s(x) = c\frac{M_0 s_1}{\Gamma(3/2)}\frac{\sqrt{x}}{s^{3/2}}e^{-\frac{x}{s}} + \frac{M_0(s_2 - cs_1)}{\Gamma^2(3/2)}\frac{\sqrt{x}}{s^{3/2}}e^{-\frac{x}{s}} \times \sum_{n=1}^{\infty}\frac{(-1)^n}{c^n}\frac{\Gamma(\nu n + 3/2)}{\Gamma(vn+1)s^{vn+3/2}}\,{}_1F_1\left(-vn;\frac{3}{2};\frac{x}{s}\right). \quad (26)$$

Here, c is an arbitrary constant which must be determined from the normalization condition. Taking the relation $F(x) = \lim_{s\to 0} F_s(x)$ into account and using the asymptotic expansion of the confluent hypergeometric function, we obtain the asymptotics of the distribution function in the inertial interval in the following form:

$$F(x) \propto \frac{M_0(s_2 - cs_1)}{\Gamma(3/2)}\left(-\frac{1}{c}\right)\frac{\Gamma(v+3/2)}{\Gamma(v+1)\Gamma(-v)}\frac{1}{x^{v+1}} = A\frac{1}{x^{v+1}}. \quad (27)$$

Thus, we have obtained the asymptotics of the stationary distribution function in the inertial intervals with different directions of the flow in the phase space. It is worth noting that the power characters of its behavior in these inertial intervals are different. In this case, the exponent is determined by a magnitude of the flow (and by its sign) and serves as an example of the new type of universality for stationary nonequilibrium systems. As seen from Eq. (24), $v \xrightarrow[P\to 0]{} 1$. Nullifying the flow leads to equilibrium distributions, and the different signs of the flow correspond to the cases where the index $v > 1$ and $v < 1$.

Above, we have analyzed only the final form of the distribution functions formed as a result of the evolution of the systems of particles with sources and sinks of particles or energy. We now consider the evolution of the formation of nonequilibrium stationary states with steady flows in the phase space. As usual, the coupling between the distribution function and a flow is assumed in the form of a relation local in time, i.e., the distribution function and the flow are referred to in the same time moment.

At the great intensities of sources and hence, at the strong nonstationarity, it is necessary to account for the characteristic duration of the relaxation of a flow of particles τ. In this case, we should consider the relaxation of the flow between collisions. Then the kinetic equation for the isotropic distribution function takes the form of a system of equations of the hyperbolic diffusion in the energy space:

$$\frac{\partial f}{\partial t} = -\frac{1}{4\pi v^2}\frac{\partial}{\partial v}J\{f,v\} + \Psi(v)\,; \quad \tau\frac{\partial J}{\partial t} + J = \Pi(v,\{f(v)\}). \quad (28)$$

As usual, the formulas for the flows of interacting particles contain integro-differential operators, and the study of the processes of evolution of the distribution function becomes quite a complicated problem.

The analysis of the time evolution of a distribution function can be performed in a significantly simpler way, if the kinetic equation has the form of a differential equation. Such physical cases are well-known: for example, we mention the interaction of photons and electrons (the Kompaneets equation for Compton processes[27]). In the next section, we will analyze this equation.

3.3. *Evolution of the distribution functions to the stationary nonequilibrium solutions of a kinetic equation in the differential form*

We now consider the formation of nonequilibrium quasistationary states of photons which interact with the free gas of electrons at a temperature T and possess the distribution functions $f(v)$ over frequencies v. It is convenient to introduce the dimensionless energy of a boson, x, instead of the frequency and the dimensionless time: $x = (hv/mc^2)$, $t = \tau v_0$, $\alpha = T/mc^2$, $v_0 = \sigma_T n_e c$. Then the Kompaneets equation can be written in these variables as the hyperbolic system

$$\frac{\partial f}{\partial t} = -\frac{1}{x^2}\frac{\partial J}{\partial x} + \psi(x)\,, \quad \tau_J \frac{\partial J}{\partial t} + J = \Pi(x, \{f(x)\})\,,$$
$$\Pi(x, \{f(x)\}) = -x^4\left(\alpha\frac{\partial f}{\partial x} + f + f^2\right). \tag{29}$$

We assume that the sources and sinks are localized near certain frequencies $v_{i,s}$ (here, $v_{i,s}$ are the characteristic frequencies of a source and a sink, respectively). The stationary nonequilibrium solution of the kinetic equation, which nullifies the collision integral, can be realized in the region of frequencies between a source and a sink, i.e., $v_i < v < v_s$, and is determined from the equation

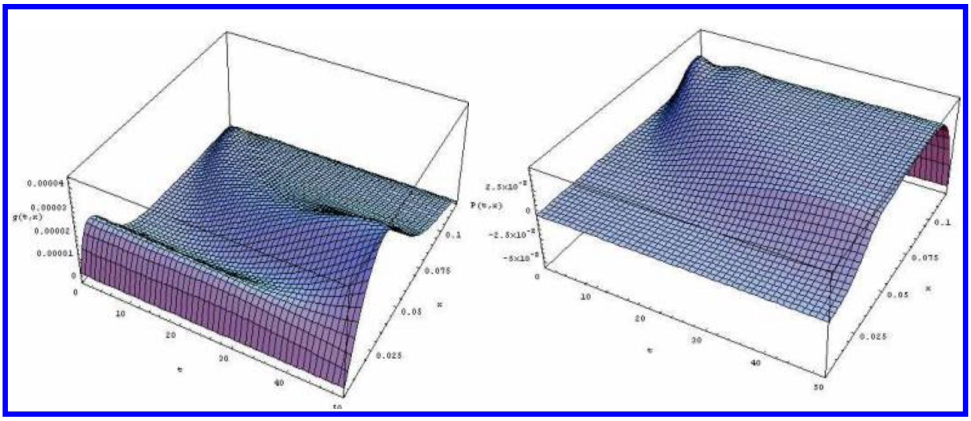
$$\Pi(x, \{f(x)\}) = -\int_0^x dx' x'^2 \Psi(x')\,. \tag{30}$$

The solution of Eq. (30) determines the nonequilibrium distribution with the steady flow of a number of particles in the energy space and has, as shown in Ref. 28, power asymptotics. In Fig. 1, we present the results of the numerical analysis of the evolution to these distributions.

We have assumed that the initial distribution is the equilibrium Planck one, and the δ-like source of quanta is positioned in the region of high frequencies. In Figs. 1(a) and 1(b), we show, respectively, the evolutions of the distribution function and the flow.

It is well-seen that the formation of a stationary distribution function of photons over the frequency occurs through the formation of the regions of frequencies, in which the flow becomes almost independent of the frequency. In these regions, the distribution functions have power character. We see the trajectories of perturbations of the distribution function, which start from the source and form the very distribution function.

The evolution shown in these figures is characteristic of many other physical situations. For example, one observes a similar evolution of phonons and electrons in solids.



(a) (b)

Fig. 1. Evolution of the dependence of the distribution function of photons and their flow on the frequency. In Fig. 1(a), we present the distribution function multiplied by the density of states of the photon gas $g(x,t) = x^2 f(x,t)$.

Using the above-given numerical solutions of the Kompaneets kinetic equation with a monochromatic source, we can show that the evolution results in the formation of the distribution function of photons which can be represented in the analytic form involving the quasipower function $\exp_q(x)$:

$$f_q(\varepsilon) = \frac{1}{\exp_q\left(\dfrac{\varepsilon - \mu}{T}\right) - 1}\,. \tag{31}$$

This solution is a generalization of the Planck function with chemical potential μ to the nonequilibrium case, and its parameters depend on the magnitude of the flow of quanta in the phase space. The plots of the distribution function of photons for various magnitudes of the flow are shown in Fig. 2. The distribution function of photons over frequencies is positioned above the Planck equilibrium distribution function, if the flow is directed toward the region of high energies. If the flow in the phase space is directed toward low energies, then the nonequilibrium distribution function has smaller values than the equilibrium one.

We note that the Kompaneets equation can describe not only photons. Equations of the Kompaneets type describe, for example, phonons in solids.

We mention the application of this equation to the description of the kinetics of pions in the nuclear matter. On the level of scales of the energies and the masses of hadrons (≈ 1 GeV), we can neglect the pion mass and describe the electromagnetic phenomena of the pion-nucleus interaction in the "soft" limit.

In this case, the hyperbolic character of the equations yields the possibility for shock waves to arise in the spectrum of pions analogously to the solutions obtained for photons in Ref. 29.

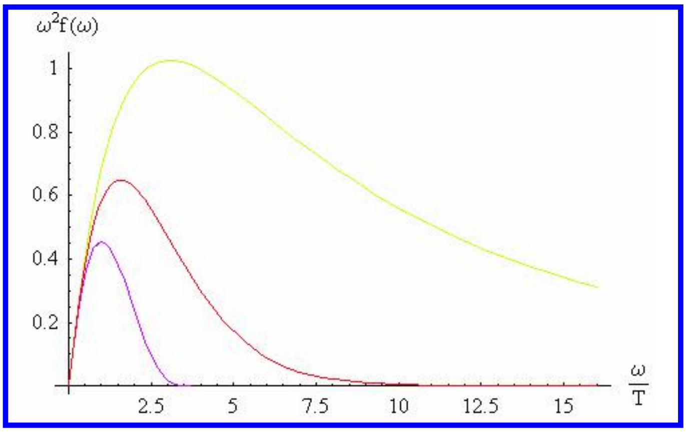


Fig. 2. (Color online) Distribution functions of photons for various flows in the phase space. The central curve corresponds to the Planck distribution function (the zero flow). The curves are positioned above and below of the equilibrium distribution depending on the flow direction: toward the increase in energy or in the opposite direction.

4. Oscillatory Processes in Quasistationary Nonequilibrium States

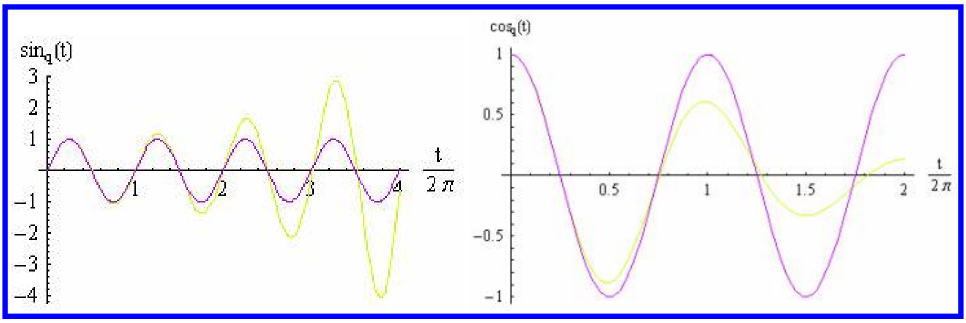
The Tsallis thermostatics is constructed on a formal generalization of the exponential and logarithmic functions on the basis of the power-like functions (10). In terms of these functions, the main thermodynamic quantities including, first of all, entropy are expressed. The power character of the used generalizations is related to the properties of similarity and nonlocality of the systems in nonequilibrium states.

The nonextensive nonideal states of a system must reveal themselves, naturally, also in the oscillatory processes which are realized in the systems with such quasistationary states. To study the peculiarities of these processes, we introduce a generalization of trigonometric functions on the basis of the q-exponential functions (10). The new functions are

$$q\,\cos(z) = \frac{\exp_q(i\,z) + \exp_q(-i\,z)}{2}\,; \quad q\,\sin(z) = \frac{\exp_q(i\,z) - \exp_q(-i\,z)}{2i}\,. \tag{32}$$

These functions pass in ordinary trigonometric functions as $q \to 1$. However, the greater the deviation of the nonextensiveness parameter q from unity, the greater the deviation of the behavior of these functions from that of the ordinary functions. Moreover, the character of the deviation is significantly different for $q < 1$ and $q > 1$. In Fig. 3, we show oscillations in the media with $q > 1$ and $q < 1$. It is seen that the medium for $q > 1$ is passive (oscillations decay), whereas the medium for $q > 1$ is active (oscillations are strengthened). Such simplest oscillatory modes can be described by solutions of the ordinary equation for oscillations with decay or instability $(d^2x/dt^2) + 2\delta(dx/dt) + \omega^2 x = 0$. This equation has a general solution in the form of decaying oscillations:

$$x = e^{-\delta t}\sin\left(\sqrt{\omega^2 - \delta^2}\,t + \Delta\varphi\right). \tag{33}$$



(a) $-q > 1$ (b) $-q < 1$

Fig. 3. (Color online) Oscillations in a nonequilibrium medium.

By comparing the linear oscillations and the oscillations in a nonequilibrium medium and taking the condition of the maximum closeness of trajectories (for example, in the sense of the quadratic norm), we can obtain the connection of the nonextensiveness parameter q and the instability increment or the damping decrement of linear oscillations.

As was noted at the beginning of the present work, the nonequilibrium states are closely related to the nonlocality (and, hence, to the memory effects) and can be described with fractional integral operators.[19,20] The influence of the memory effects is manifested in the system in that the external forces $F(t)$ acting on the system cause the origin of a flow of particles:

$$J(t) = \int_0^t dt' M(t-t') F(t') \,, \tag{34}$$

where $M(t-t')$ is the memory function. When the memory function has the character of the δ-function, then the system has no memory. In this case, the flow at the time moment t is determined by the force at the same time moment. On the power behavior of the memory function $M(t-t') \propto (t-t')^{\alpha-1}$, the memory manifests itself, and the coefficient $0 < \alpha < 1$ characterizes the degree of conservation of the memory in the system ($\alpha = 1$ corresponds to the ideal memory). In this case, the flow can be written in terms of the fractional integrals which are a natural generalization of the iterated integral in the n-dimensional space (the Cauchy formula),

$$I^n g(x) \stackrel{\text{def}}{=} \int_{x_0}^{x} \int_{x_0}^{x_{n-1}} \cdots \int_{x_0}^{x_1} g(\xi) d\xi dx_1 \cdots dx_{n-1} = \frac{1}{(n-1)!} \int_{x_0}^{x} \frac{g(\xi)}{(x-\xi)^{1-n}} d\xi \,,$$

and are given by the formula[28]

$$I^v g(x) \equiv D^{-v} g(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(v)} \int_{x_0}^{x} \frac{g(\xi)}{(x-\xi)^{1-v}} d\xi \,. \tag{35}$$

Finally, we write

$$J(t) = J_0 D^{-\alpha} F(t) \,. \tag{36}$$

The integral operator determining the flow in Eq. (36) can be considered as the regularizing operator for a force acting on particles[29] and, simultaneously, as the averaging operator which takes the character of the action into account in a self-consistent way. As mentioned in Ref. 30, the memory significantly affects the oscillatory process. Indeed, let us consider the oscillations in a medium with memory. The equation with fractional derivatives for oscillations in the medium with the memory coefficient α looks like

$$D^{2-\alpha}q + \omega^2 q = 0\,, \tag{37}$$

and its exact solution can be written in terms of the Mittag–Leffler function as

$$q(t) = c_1 t E_{2-\alpha,2}(-\omega^2 t^{2-\alpha}) + c_2 E_{2-\alpha,1-\alpha}(-\omega^2 t^{2-\alpha})\,. \tag{38}$$

The analysis of this solution allows us to conclude that the influence of the memory effects, as well as that of the effects arising in nonequilibrium states, is equivalent to the effective damping. This is seen from the comparison of the phase portraits of oscillations in a nonequilibrium medium and in a medium with memory shown in Fig. 4.

By optimizing the difference of phase trajectories by the method of least squares, we obtain the dependence of the nonextensiveness parameter q on the memory parameter α:

$$q(\alpha) = 3.96 - 4.00\alpha + 1.05\alpha^2\,. \tag{39}$$

In connection with the fact that the nonequilibrium states and the memory effects lead to power functions, we may assert that the properties of similarity join all the effects under consideration. The effects of similarity, from our viewpoint, are most naturally described in the framework of quantum analysis.

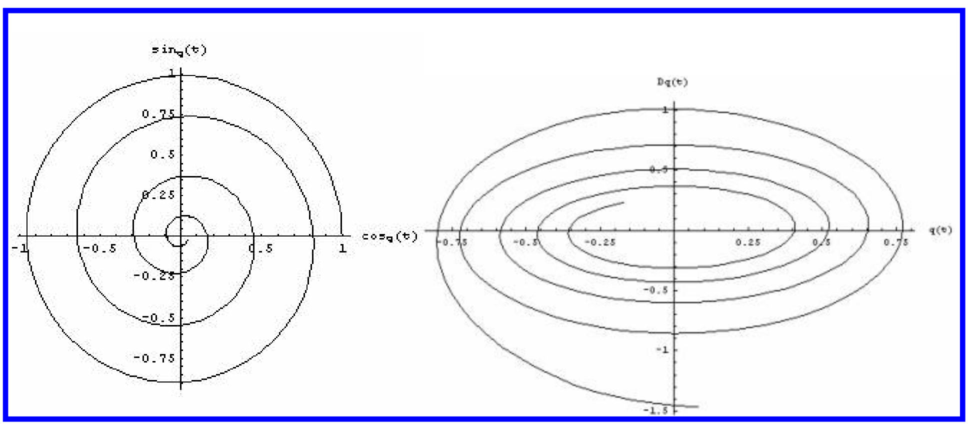


Fig. 4. Phase trajectories of oscillations in a nonequilibrium medium and in a medium with memory.

5. Quantum Analysis and Nonequilibrium States

The ordinary derivatives are customarily introduced on the basis of shear transformations of the argument, therefore, the effects of similarity enter into such differential calculus not quite naturally. In quantum analysis, this drawback is overcome with the use of Jackson operators[21] (see also Refs. 34 and 35). The merit of quantum analysis consists in that the q-derivatives (Jackson derivatives) are introduced on the basis of a scale transformation of the argument (with the similarity coefficient q):

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x} \,. \tag{40}$$

In the limit case $q \to 1$, the Jackson derivative passes in the ordinary derivative $D_x^1 f(x) = \lim_{q\to 1} D_q f(x)$. As for the quantum q-derivative of a power function, it is calculated in a very simple way: $D_q x^n = ((qx)^n - x^n/(q-1x) = (q^n - 1/q - 1)x^{n-1} = [n]x^{n-1}$. Here, we have introduced q-numbers $[\alpha]_q = (q^\alpha - 1/q - 1)$ which are related to the q-generalization (10) of a logarithm: $\ln_q(x) = (x^{1-q} - 1/1-q) = (x - 1([1-q]_x/1-q)$. It is also simple to find the derivative of a function possessing the property of similarity. Let $f(qx) = q^\alpha f(x)$, then

$$D_q f(x) = \frac{(q^\alpha f(x)) - f(x)}{(q-1)x} = \frac{q^\alpha - 1}{q - 1}\,\frac{f(x)}{x} = [\alpha]\frac{f(x)}{x} \,.$$

In quantum analysis, one widely uses the q-generalization of an exponential function:

$$e_q^x = \sum_{k=0}^{\infty} \frac{x^n}{[k]!} \,. \tag{41}$$

The difference consists in the change of the ordinary function $k!$ by the symbol $[k]!$ such that

$$[k]! = 1\,, \quad \text{if} \;\; k = 0 \quad \text{and} \quad [k]! = [k][k-1]\cdots[k]\,, \quad \text{if} \;\; k \geq 1\,. \tag{42}$$

It is easy to see that such a definition leads to the fact that the function e_q^x is the eigenfunction of the operator D_q:

$$D_q e_q^x = D_q\left(\sum_{k=0}^{\infty} \frac{x^k}{[k]!}\right) = \sum_{k=0}^{\infty} \frac{1}{[k]!} D_q\left(x^k\right) = \sum_{k=1}^{\infty} \frac{[k]}{[k]!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{[k-1]!} x^{k-1} = e_q^x \,. \tag{43}$$

The generalized exponential function e_q^{-x} tends to the ordinary one as $q \to 1$, and decreases more slowly with increase in x for $q < 1$ than the ordinary exponential function, which is shown in Fig. 5.

The quantum derivative is a linear operator. Therefore, the q-derivative of a linear combination of functions is expressed through the derivatives of separate

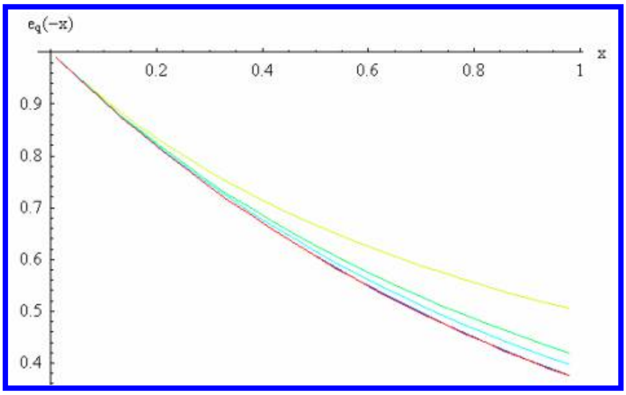


Fig. 5. (Color online) Plots of the exponential function and its q-generalization for different values of q:

- Upper curve — for the function e_q^x for $q = 0.001$.
- Lower curve — for the exponential function.

functions by the ordinary formula. However, the q-derivative of a product of functions has its own peculiarities. Definition (39) directly yields the relations which differ from the ordinary ones by the asymmetry:

$$\begin{aligned} D_q(f(x)g(x)) &= f(qx)D_q g(x) + g(x)D_q f(x) \\ D_q(f(x)g(x)) &= f(x)D_q g(x) + g(qx)D_q f(x)\,. \end{aligned} \tag{44}$$

For the functions possessing the similarity, i.e., $f(qx) = q^\alpha f(x)$ and $g(qx) = q^\beta g(x)$, we get:

$$\begin{aligned} D_q(f(x)g(x)) &= f(x)D_q g(x) + q^\beta g(x)D_q f(x) \\ &= f(x)D_q g(x) + g(x)D_q f(x) + (q^\beta - 1)g(x)D_q f(x)\,. \end{aligned} \tag{45}$$

It follows from this relation that the similarity parameter of the quantum differentiation, q, simultaneously characterizes the degree of its asymmetry.

Let us consider the peculiarities of oscillatory processes in media with regard to their similarity. To this end, we consider the q-trigonometric functions

$$\cos_q(z) = \frac{e_q^{iz} + e_q^{-iz}}{2}; \quad \text{and} \quad \sin_q(z) = \frac{e_q^{iz} - e_q^{-iz}}{2i} \tag{46}$$

which are the solutions of the equation for oscillations given in the form $D_q(D_q f(x)) + f(x) = 0$. The phase trajectories for this equation in the cases where the nonextensiveness parameter $q < 1$ and $q > 1$, are shown in Fig. 6.

It is seen that the phase portraits of q-oscillators are similar to those for decaying or unstable oscillations. We may say that the fractality of the medium leads to its nonideality (the medium becomes either passive or active). The character of oscillatory processes depends on the similarity parameters and, with regard to the above-mentioned couplings (see Sec. 3.3), on the direction of flows in the phase space.

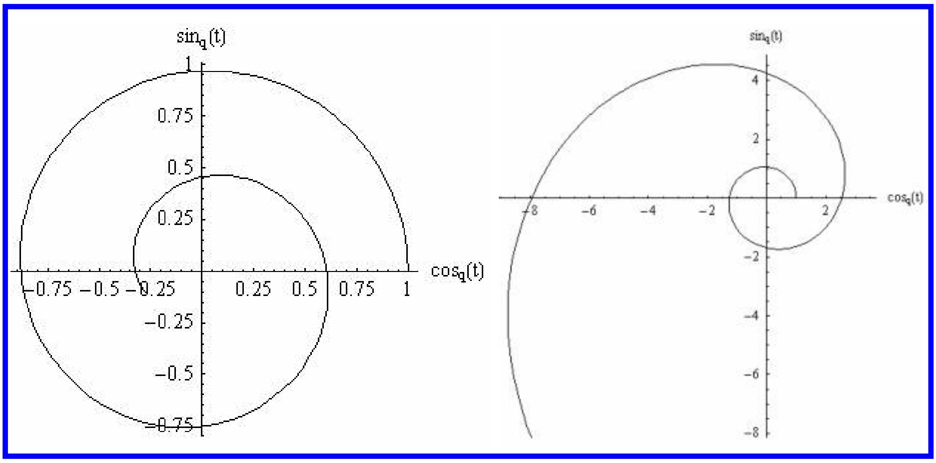


Fig. 6. Phase trajectories for the oscillatory process in a medium with similarity.

6. Conclusion

The self-organization of a system of particles is related, naturally, to the growth of structures. However, in the framework of nonequilibrium thermodynamics, one usually considers the growth of structures in the coordinate space. In this case, homogeneous systems of particles are analyzed as equilibrium systems without structures and flows. However, the structures in spatially homogeneous systems can appear not only in the configurational space, but also in the momentum one. The analysis of a growth of structures in the whole phase space is a very complicated problem. To solve it, we propose the following approximate approach including two stages. On the first stage, we consider a system of particles which is homogeneous in the configurational space and undergoes the action of spatially homogeneous sources of energy or particles. Such an action is quite analogous to the action of a mass force in theoretical mechanics, and leads to the appearance of the flow in the momentum space (the flow of particles or energy in the energy space).

The flows in the momentum space lead to nonlocal nonextensive states of the system of interacting particles. On the second stage, it will be necessary to consider the formation of spatially inhomogeneous structures in systems with locally nonequilibrium states of particles, which are determined on the first stage.

In the framework of the first stage of the analysis of the processes of self-organization, we have analyzed the various approaches to the description of properties and conditions of the formation of nonextensive states in systems of many particles on the basis of kinetic equations with different forms of the collision integral:

- For classical statistics, it is shown that there exist the power-like functions which nullify the Lenard–Balescu collision integral and the Boltzmann collision integral for Maxwell molecules in spatially homogeneous systems of particles;

- We have demonstrated the appearance of power asymptotics of the distribution functions of Maxwell molecules with the exponents, depending on the magnitude and the direction of flows in the phase space;
- We have proposed and analyzed the Kompaneets hyperbolic equation with regard to the relaxation of the flow of photons in the frequency space.

By using the Kompaneets hyperbolic equation, we have demonstrated the formation of the power-like distribution functions of photons with regard to the sources of gamma-quanta, which leads to the appearance of the flow of quanta in the frequency space of the system of photons. The results of numerical calculations of the evolution show that the power asymptotics arise in those frequency regions where the flow turns out to be independent of the frequency of quanta. The condition for locally nonequilibrium distribution functions to be formed is the condition for the source power Q_i ensuring the excess of the formed nonequilibrium flow over the flow of particles tending to establish the equilibrium distribution function with the effective density $n_0(Q_i > (n_0/\tau))$.

We have presented the analogy between photons and pions in the "soft" limit (when the ratio of the masses of a pion and a nucleon tends to zero), in order to study the evolution of the distribution function of pions in a nuclear system with regard to the processes of photoproduction of pions.

We have obtained the connection between the nonextensiveness parameter and the similarity parameter using the quantum q-generalization of the exponential function, which ensures the qualitative coincidence of the behaviors of the Tsallis distribution function and the generalized quantum exponential function.

We have shown that the oscillatory processes in nonequilibrium nonconservative media can be described by the equations for oscillations with the use of Jackson derivatives or fractional derivatives instead of ordinary ones. We have obtained the dependence of the damping increment or decrement on the exponent of a fractional derivative or the similarity coefficient in the Jackson derivative. That is, it is possible to change the operators,

$$\frac{\partial}{\partial t} + \frac{1}{\tau} \to D^{\alpha} \quad \text{or} \quad \frac{\partial}{\partial t} + \frac{1}{\tau} \to D_q \,, \tag{47}$$

by conserving the qualitative behavior of phase trajectories of the system. These effects cause the appearance of the derivatives, whose order depends on the fractality or nonconservativeness of the medium, in kinetic equations. For example, the Kompaneets-type differential kinetic equations (29) for the evolution of the distribution function of bosons ($\delta = 1$) or fermions ($\delta = -1$) are transformed to the form

$$\begin{aligned} g(\varepsilon)\frac{\partial f(\varepsilon,t)}{\partial t} &= -\frac{\partial^{\alpha}}{\partial \varepsilon^{\alpha}}\left[j(\varepsilon,t)\right] + g(\varepsilon)\psi(\varepsilon), \\ \tau\frac{\partial^{\beta} j(t,\varepsilon)}{\partial t^{\beta}} &= D(\varepsilon)\left(T\frac{\partial^{\alpha} f}{\partial \varepsilon^{\alpha}} + f(1+\delta f)\right). \end{aligned} \tag{48}$$

These equations can become the basis of the kinetic theory of fractal media. The solutions of the fractional integro-differential equations (48) have the power asymptotics which correspond to the well-known modes of sub- or superdiffusion.

From our viewpoint, the use of relations (47) will allow one to formulate, in the future, both the Hamilton equations by applying fractional operators (or quantum analysis) for a system of particles in the nonconservative case, and the relevant variational principle. The further study of the growth of structures in the framework of the second stage can be carried out by using the Vlasov covariant kinetic equation with regard to the self-consistent fields in locally nonequilibrium states. One of the most important applications of this theory to the study of the evolution of structures in nuclear physics under a modification of Coulomb barriers in nonequilibrium states will be analyzed in the subsequent work.

References

1. G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems. From Dissipative Structures to Order through Fluctuations* (Wiley, New York, 1977).
2. NuPECC (Expert Committee of the European Science Foundation). Long Range Plan 2004: Perspectives for Nuclear Physics Research in Europe in the Coming Decade and Beyond.
3. W. Greiner, Exotic nuclei: from super-heavies to hyper and antimatter, *Nuclear Physics* **64**(6), 1191 (2001).
4. V. G. Boiko, L. L. Jenkovszky and V. M. Sysov, Thermodynamics of phase transitions in nuclear matter, *Fiz. Elem. Chast. Atom. Yadr.* **22**(3), 675 (1999).
5. T. Maruyama, K. Niita, K. Oyamatsu *et al.*, Quantum molecular dynamics approach to the nuclear matter below the saturation density, *Phys. Rev. C* **57**(2), 655 (1998).
6. S. V. Adamenko, F. Selleri and A. van der Merwe, *Controlled Nucleosynthesis. Breakthroughs in Experiment and Theory*, Series: Fundamental Theories of Physics, Vol. 156 (Springer, Dordrecht, Netherlands, 2007).
7. S. V. Adamenko and V. I. Vysotskii, Neutronization and protonization of nuclei: two possible ways of the evolution of astrophysical objects and the laboratory electron-nucleus collapse, *FoPL* **19**(1), 21 (2006).
8. S. V. Adamenko, A. S. Adamenko, A. A. Gurin and Yu. M. Onishchuk, Track measurements of fast particle streams in pulsed discharge explosive plasma, *Radiation Measurements* **40**(2–6), 486 (2005).
9. S. Adamenko, V. Novikov, A. Pashchenko and I. Shapoval, *Uspekhi Sovremen. Radioel.* **11**, 55 (2005).
10. N. N. Bogolyubov, *Problems of Dynamical Theory in Statistical Physics* (Gostekhizdat, Moscow, 1946).
11. A. A. Vlasov, *Nonlocal Statistical Mechanics* (Nauka, Moscow, 1978).
12. V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations* (Dover, New York, 1982).
13. E. Cartan, *Les espaces de Finsler* (Actualities, Paris, 1934).
14. A. N. Gordeev, Power energy spectrum of cosmic rays as a solution of the general-covariant kinetic equation, *Astr. Zh.* **43**, 1030 (1966).
15. A. V. Kats, V. M. Kontorovich, S. S. Moiseev and V. E. Novikov, Power solutions of the Boltzmann kinetic equation describing the distribution of particles with flows over the spectrum, *Pis'ma Zh. Eksp. Teor. Fiz.* **21**, 13 (1975).

16. V. I. Karas, V. E. Novikov and S. S. Moiseev, Exact power solutions of kinetic equations in solid-state plasma, *Zh. Eksp. Teor. Fiz.* **71**, 744 (1976).
17. S. I. Kononenko, V. I. Karas, V. E. Novikov, R. Z. Sagdeev *et al.*, Nonequilibrium distribution functions of electrons in the plasma of a semiconductor irradiated by fast ions, *Fiz. Plazm.* **30**(8), 1 (2004).
18. C. Tsallis, Nonextensive thermostatics: brief review and comments, *Physica A* **221** 277 (1995).
19. M. M. Dzhrbashyan, *Integral Transformations and Representations of Functions in the Complex Region* (Nauka, Moscow, 1966).
20. K. Kolwankar and A. Gangal, Fractional differentiability of nowhere differentiable functions and dimensions, *Chaos* **6**, 505 (1996).
21. V. Kac and P. Cheung, *Quantum Calculus* (Springer, Berlin, 2002).
22. F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E* **55**(3), 3581 (1997).
23. M. Kac, *Probability and Related Topics in Physical Sciences* (Interscience, London, 1957).
24. R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, New York, 1975).
25. A. V. Bobylev, *Dokl. AN SSSR* **225**(6), 1296 (1975).
26. M. H. Ernst, Nonlinear model — Boltzmann equations and exact solutions, *Physics Reports* **78**(1), 1 (1981).
27. A. S. Kompaneets, The kinetic equation for photons, *Zh. Eksp. Teor. Fiz.* **31**, 876 (1956).
28. V. E. Novikov, S. S. Moiseev, A. V. Tur and V. V. Yanovsky, Universal nonequilibrium distributions of photons and the dynamics of their formation, *Zh. Eksp. Teor. Fiz.* **86**(3), 920 (1984).
29. Ya. B. Zel'dovich and R. A. Syunyaev, The structure of a shock wave in the emission spectrum on the Bose-condensation of photons, *Zh. Eksp. Teor. Fiz.* **62**(1), 153 (1972).
30. A. I. Olemskoi and V. F. Klepikov, Theory of spatiotemporal pattern in non-equilibrium systems, *Phys. Rep.* **338**(6), 571 (2001).
31. V. N. Bolotov and V. E. Novikov, Description of chaotic dynamical system by regularization-based methods, in *Proc. 7-th Int. Conf. CryMiKo'97*, Sevastopol, Crimea, Ukraine, 15–18 September 1997 (Weber Co., Ukraine, 1997), pp. 264–265.
32. S. Kruchinin, V. Klepikov and V. E. Novikov, Nonlinear current oscillations in a fractal Josephson junction, *Mater. Sci.* **23**(4), 1009 (2005).
33. F. Jackson, *Mess. Math.* **38**, 57 (1909).
34. V. Kac and P. Cheung, *Quantum Calculus* (Springer, Berlin, 2002).
35. V. Klepikov, S. Kruchinin, V. Novikov and A. Sothikov, Composite materials with radioactive inclusions as artificial radio absorbing covering, *Rev. Adv. Mater. Sci.* **11**, 34 (2006).