New Trends in Superconductivity

Edited by

James F. Annett and Sergei Kruchinin

NATO Science Series

II. Mathematics, Physics and Chemistry – Vol. 67

CALCULATION OF CORRELATION FUNCTIONS FOR SUPERCONDUCTIVITY MODELS

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We begin with a discussion on the approximating Hamiltonian method in the case of four-fermion interaction. Namely, we consider a general class of models with four-fermion pair interaction for which an asymptotically exact solution can be constructed. A definition of the asymptotically exact solution as well as methods of its construction for this type of models was formulated by N.N. Bogolubov, D.N. Zubarev and Y.A. Tserkovnikov in [1, 2, 3, 4]. In these works an approximating procedure was invented, into which the ideas of a method based on the introduction of the so-called "approximating Hamiltonian" were incorporated, as well as the reasons were given to believe that the solution obtained for this approximating Hamiltonian might become exact upon the standard thermodynamical limit of an infinitely large system $V \to \infty$. In [1, 2] the following model Hamiltonian was brought to attention

$$H = H_0 + H_{int}, \tag{1}$$

$$H_0 = \sum_{(p,s)} (E(p) - \mu) a^{\dagger}_{ps} a_{ps}, \;\; H_{int} = -\frac{1}{V} \sum_{(p,p')} J(p,p') a^{\dagger}_{-p,-\frac{1}{2}} a^{\dagger}_{p,\frac{1}{2}} a_{p',\frac{1}{2}} a_{-p',-\frac{1}{2}},$$

where $a^{\dagger}_{p,\pm\frac{1}{2}}$ and $a_{p,\pm\frac{1}{2}}$ are the Fermi operators and V is the volume of the system. The kernel $J(p,p')$ is assumed to be a real bounded function vanishing outside of a certain domain of momenta. The summation in H_{int} is performed over the momenta p and p' belonging to the energy layer $E_F - \omega < E(p) < E_F + \omega$. It was shown that for such a Hamiltonian it is possible to construct an expression for the free energy which is asymptotically exact in the limit $V \to \infty$. The main idea of this method consists in introduction of the so-called "trial Hamiltonian" $H_0(C)$ being a quadratic in Fermi operators form that depends in addition on some arbitrary constants C. Such a Hamiltonian can be easily diagonalized so that one can calculate the free energy corresponding to it. In [1] reasons were put forward to assume that in the limit $V \to \infty$ the approximating free energy $F_0(C)$ becomes equal to the initial free energy F calculated with respect to the original Hamiltonian H. This result was obtained by means of the perturbation theory. The reasoning was based on the fact that in the perturbation theory series all subsequent terms representing corrections to this solution are asymptotically small in the limit $V \to \infty$. However, the convergence of the perturbation theory series was not investigated. In later work [2] the model (1) was treated without recourse to any perturbation theory. In this work a chain of coupled equations for Green's functions was studied and it was shown that the Green's functions calculated for the exactly solvable model, that one represented by the approximating Hamiltonian $H_0(C)$, satisfy the whole chain of equations for the original Hamiltonian H with an error of order $(1/V)$.

From a purely mathematical point of view, however, it is obvious that reasoning of this type are not entirely satisfactory. Nevertheless, the above-mentioned results made a substantial contribution to the justification of the approximating Hamiltonian method and investigation of the limiting properties of the asymptotically exact solutions. We ought to note that the mathematically rigorous proof of results obtained in [2] meets substantional mathematical difficulties in general case. For a special case of zero temperature this proof was given by Bogolubov [3]. It was found that the approximating solution exists for the model (1) and becomes asymptotically exact as V tends to infinity. The ground state energy and the Green's functions were also calculated exactly in the same limit by means of a special majoritating technique.

Unfortunately, direct generalization of the results of [3] for the case of arbitrary temperature $\theta \geq 0$, though being of considerable interest from many points of view, was found to be impossible because the method of proof in this work was inherently limited to the treatment of the ground state. Later on, the same complicated problem of rigorous mathematical proof of the validity of the approximating Hamiltonian method in the case of non-zero temperatures was addressed for some model Hamiltonians in our works [5, 6].

Let us consider a model dynamical system based on Fermi operators [7, 9, 13, 14]

$$H = \sum_{(f)} T_f a_f^\dagger a_f - \frac{1}{2V} \sum_{(f,f')} \lambda(f)\lambda(f') a_f^\dagger a_{-f}^\dagger a_{-f'} a_{f'}. \qquad (2)$$

We use notations $f = (p, s)$ and $-f = (-p, -s)$ to denote a set of quantum numbers of a fermion: the momentum p and the spin projection s. Here

$$p_x = \frac{2\pi n_x}{L},\ p_y = \frac{2\pi n_y}{L},\ p_z = \frac{2\pi n_z}{L}, \quad n_x, n_y, n_z \quad \text{are integers}, \quad V = L^3,$$

and $T_f = \frac{p^2}{2m} - \mu$ where μ is the chemical potential. For the standard BCS model it is generally accepted that

$$\lambda(f) = \begin{cases} J\varepsilon(s) = \text{const}, & \left|\frac{p^2}{2m} - \mu\right| \le \Delta_\varepsilon, \quad \varepsilon(s) = \pm 1 \\ 0, & \left|\frac{p^2}{2m} - \mu\right| > \Delta_\varepsilon \end{cases} \qquad (3)$$

But in this article we do not use these concrete properties of the functions $\lambda(f)$ and T_f. Instead, it is sufficient to impose the following general restrictions on them. The functions $\lambda(f)$ and T_f are assumed to be real and $\lambda(-f) = -\lambda(f)$. We also assume that

$$\frac{1}{V}\sum_{(f)} |\lambda_f| \le K_1 = \text{const}, \qquad \frac{1}{V}\sum_{(f)} |T_f \lambda_f| \le K_2 = \text{const},$$

$$\frac{1}{V}\sum_{(f)} \lambda_f^2 \le K_3 = \text{const} \quad \text{if} \quad V \to \infty. \qquad (4)$$

These conditions are certainly fulfilled in particular case (3).

Let us rewrite Eq.(2) as

$$H = H_0 + H_1$$

where the approximating Hamiltonian H_0 is given by

$$H_0 = H = \sum_{(f)} T_f a_f^\dagger a_f - \left\{ \sum_{(f)} \left(C a_{-f} a_f + C^* a_f^\dagger a_{-f}^\dagger \right) \right\} + 2VC^*C \qquad (5)$$

and

$$H_1 = 2V \left(\frac{1}{2V} \sum_{(f)} \lambda_f a_f^\dagger a_{-f}^\dagger - C \right) \left(\frac{1}{2V} \sum_{(f)} \lambda_f a_{-f} a_f - C^* \right),$$

C and C^* being c-numbers. Since H_0 is a quadratic form in the Fermi operators it can be diagonalized by the $u - v$ transformation

$$a_f = u(f)\alpha_f - v(f)\alpha_f^\dagger$$

and the corresponding free energy per unit volume can be calculated:

$$f_{H_0} = -\frac{\theta}{V} \ln Sp\, e^{-\frac{H_0}{\theta}}$$

The value of the complex parameter C in the trial Hamiltonian (5) must provide the absolute minimum for $f_{H_0(C)}$

$$f_{H_0(C)} = \min$$

from which it follows that

$$\frac{\partial f_{H_0}}{\partial C} = 0, \qquad C = \langle J \rangle_{H_0} = \frac{SpJe^{-\frac{H_0}{\theta}}}{e^{-\frac{H_0}{\theta}}} \tag{6}$$

where

$$J = \frac{1}{2V} \sum_{(f)} \lambda(f) a_f^\dagger a_{-f}^\dagger.$$

We constructed a method which allows to prove that the difference $f_{H_0} - f_H$ of the free energies calculated on the basis of the approximating and the original model Hamiltonians is asymptotically small. For this purpose it is convenient to consider an auxiliary model system with the Hamiltonian containing sources whose strength is characterized by the parameter ν

$$\Gamma = T - 2VJJ^\dagger - (\nu J + \nu^* J)\, V, \qquad \nu = re^{i\phi}. \tag{7}$$

When $\nu = 0$ then the Hamiltonian (7) coincides with H if

$$\sum_{(f)} T_f a_f^\dagger a_f.$$

An appropriate trial (approximating) Hamiltonian is given by

$$\Gamma_0 = T - 2V \left(CJ^\dagger + C^* J \right) - V \left(\nu J + \nu^* J^\dagger \right) + 2V|C|^2. \tag{8}$$

It is obvious that $\Gamma = \Gamma_0 + \mathcal{U}$ where

$$\mathcal{U} = -2V\,(J - C)\,(J^\dagger - C^*). \tag{9}$$

Now, let us calculate the mentioned above difference between the free energies per unit volume. For this purpose we introduce an intermediate auxiliary Hamiltonian $\Gamma_t = \Gamma_0 + t\mathcal{U}$ which coincides with the trial Hamiltonian Γ_0 if $t = 0$ and with the auxiliary Hamiltonian Γ if $t = 1$. We assume that the parameter C in the intermediate Hamiltonian is fixed and independent of t. Then we consider the statistical sum and the free energy for the intermediate Hamiltonian:

$$Q_t = Sp\, e^{\frac{\Gamma_t}{\theta}}, \qquad Q_t = -e^{\frac{Vf_t}{\theta}}, \qquad f_t(C) = -\frac{\theta}{V}\ln Q_t. \tag{10}$$

Let us notice that $f_{t=1}(C) = f_\Gamma$ and is independent of C as a consequence. Differentiating (10) twice with respect to t, we obtain

$$\frac{\partial^2 Q_t}{\partial t^2} = -\frac{V}{\theta}\frac{\partial^2 f_t}{\partial t^2}Q_t + \frac{V^2}{\theta^2}\left(\frac{\partial f_t}{\partial t}\right)^2 Q_t.$$

On the other hand, taking into account that

$$\frac{\partial^2 Q_t}{\partial t^2} = \frac{1}{\theta^2}\int_0^1 Sp\left\{\mathcal{U}e^{-\frac{\Gamma^t}{\theta}\tau}\mathcal{U}e^{-\frac{\Gamma^t}{\theta}(1-\tau)}\right\}d\tau,$$

we obtain

$$-\frac{V}{\theta}\frac{\partial^2 f_t}{\partial t^2} + \frac{V^2}{\theta^2}\left(\frac{\partial f_t}{\partial t}\right)^2 = \frac{1}{\theta^2 Q_t}\int_0^1 Sp\left\{\mathcal{U}e^{-\frac{\Gamma^t}{\theta}\tau}\mathcal{U}e^{-\frac{\Gamma^t}{\theta}(1-\tau)}\right\}d\tau,$$

where

$$\frac{\partial f_t}{\partial t} = \frac{1}{V}\frac{Sp\,\mathcal{U}e^{-\frac{\Gamma_t}{\theta}}}{Sp\,e^{-\frac{\Gamma_t}{\theta}}} = \frac{\langle\mathcal{U}\rangle_t}{V}.$$

Passing to the matrix representation in which the Hamiltonian is diagonal, we finally get an inequality

$$-\frac{\partial^2 f_t}{\partial t^2} \geq 0.$$

This inequality implies, in particular, that the magnitude

$$\frac{\partial f_t}{\partial t} \equiv \frac{1}{V}\langle\mathcal{U}\rangle_t$$

decreases as the parameter t increases. Thus we have

$$f_{\Gamma^0}(C) - f_\Gamma = -\int_0^1 \frac{\partial f_t}{\partial t} dt = -\int_0^1 \frac{\langle \mathcal{U} \rangle_t}{V} dt \geq 0.$$

Since this relation holds for arbitrary C, we have

$$\min_{(C)} f_{\Gamma_0}(C) \geq f_\Gamma, \qquad f_{\Gamma_0} \geq f_\Gamma.$$

Integrating both sides of this inequality and substituting (9) instead of $\mathcal{U}$, we can see that the following inequality holds for any C

$$f_{\Gamma_0}(C) - f_\Gamma \leq 2\langle (J - C)\,(J^\dagger - C^*)\rangle_\Gamma.$$

In particular, we can set $C = \langle J \rangle_\Gamma$ and notice that

$$f_{\Gamma_0} = \min f_{\Gamma_0}(C) \leq f_{\Gamma_0}(\langle J \rangle_\Gamma),$$

and finally

$$0 \leq f_{\Gamma_0} - f_\Gamma \leq 2\langle (J - \langle J \rangle)(J^\dagger - \langle J^\dagger \rangle)\rangle.$$

To prove that the difference $f_{\Gamma_0} - f_\Gamma$ is asymptotically small in the limit $V \to \infty$ we can apply the method developed in our work [5] so that finally we obtain an inequality [1,2]

$$0 \leq f_{\Gamma_0} - f_\Gamma \leq \frac{L}{V^{2/5}}, \qquad L = \text{const} \tag{11}$$

where L is a simple combination of the original constants (4). It is clear that this estimation is uniform with respect to $\theta \geq 0$ and therefore is valid for $\theta = 0$.

Now, let us consider an approach for the calculation of the asymptotically exact correlation functions and Green's functions for this model. This approach relies essentially on the results obtained in (11) for the free energies. In particular, we can show that

[1] This result has found numerous applications. For example, using this result, Hertel and Thirring calculated the free energy in the thermodynamic limit for a model describing a system of mutually attracting fermions [10]

[2] A similar exactly solvable model of a crystal

$$H = \sum_{(q)} T(q) a_q^\dagger a_q + \frac{V}{2} \sum_{(q)} \lambda(q) \rho_q \rho_{-q}, \qquad \rho_q = \frac{1}{V} \sum_{(k)} a_{k+q}^\dagger a_k$$

was considered by I.P.Bazarov [12]

$$|\langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_\Gamma - \langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma^0}| \le \eta\left(\frac{1}{V},\delta\right)|t-\tau| + \eta'\left(\frac{1}{V},\delta\right) \qquad (12)$$

where $\mathcal{A}, \mathcal{B} = a_f, a_f^\dagger, a_{-f}, a_{-f}^\dagger$ and

$$\eta\left(\frac{1}{V},\delta\right) \to 0, \qquad \eta'\left(\frac{1}{V},\delta\right) \to 0 \quad \text{as} \quad V \to \infty$$

for any fixed $\delta \ge 0$. We emphasize that these inequalities hold for $r \ge \delta$.

For the matter of convenience we consider the same Hamiltonian (7)

$$\Gamma = T - 2VJJ^\dagger g - r(J + J^\dagger)V \qquad (13)$$

where g is a parameter characterizing the strength of the attractive interaction (the case of the repulsive interaction has been considered in [8]) and r is a positive parameter which tends to zero in all final expressions.

The corresponding approximating Hamiltonian takes the form

$$\Gamma_0 = T - 2VCg(J^\dagger + J) + 2VC^2 g - r(J + J^\dagger)V \qquad (14)$$

here C is defined from the condition of the absolute minimum of f_{Γ_0} as usual:

$$\frac{\partial}{\partial C} f_{\Gamma_0} = 0, \quad \text{so} \quad C = \langle J\rangle_{\Gamma_0}.$$

Let us now write down equations of motion for the Hamiltonian (13)

$$\begin{aligned} i\frac{da_f}{dt} &= T_f a_f - \lambda_f a_{-f}^\dagger(2J^\dagger g + r) \\ i\frac{da_f^\dagger}{dt} &= -T_f a_f^\dagger + \lambda_f(2Jg + r)a_{-f} \end{aligned} \qquad (15)$$

It is seen that all commutators of the type $[J, J^\dagger]$, $[J, a_f]$, $[J, a_f^\dagger]$ are infinitesemal values of the order $O(1/V)$, so it is expectable that the quantum nature of variables $J, J^\dagger$ would be unimportant in the limit $V \to \infty$. Replacing this operators $J, J^\dagger$ with their "average values" C, C^*, we end up again with the approximating Hamiltonian treated in [5]. We should only notice that the equations of motion for the approximating Hamiltonian differ from the equations of motion (15) in the way that the operators $J, J^\dagger$ must be replaced by the corresponding c-numbers C, C^* in the right hand of Eqs.(15). But the operators $a_f, a_f^\dagger$ in both sets of equations coincide at the initial moment of time $t = 0$ as they must do in the Schrödinger representation. Obviously, if we want to show the proximity of the correlation functions

$$\langle \mathcal{A}(t)\mathcal{B}(\tau)\rangle_\Gamma \sim \langle \mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma_0}$$

where $\mathcal{A}, \mathcal{B} = a_f, a_f^\dagger, a_{-f}, a_{-f}^\dagger$ it will be desirable to show the proximity of the operator J to its averaged value $C = \langle J\rangle_{\Gamma_0}$. Let us consider the difference

$$d = f_{\Gamma_0} - f_\Gamma$$

of the free energies per unit volume for the Hamiltonians Γ and Γ_0 and notice that

$$\frac{\partial d}{\partial r} = \langle J + J^\dagger\rangle - 2C, \qquad \frac{\partial d}{\partial g} = 2(\langle JJ^\dagger\rangle - C^2)$$

so that

$$\langle (J - C)(J^\dagger - C)\rangle_\Gamma = \langle JJ^\dagger\rangle_\Gamma - C(\langle J + J^\dagger\rangle_\Gamma) + C^2 = \frac{1}{2}\frac{\partial d}{\partial g} - \frac{\partial d}{\partial r}C. \quad (16)$$

From the other side it is clear that

$$|J^\dagger J - JJ^\dagger| \leq \frac{K}{V}$$

where K is a constant. So, we have an estimation for the average

$$\left|\langle (J^\dagger - C)(J - C)\rangle_\Gamma\right| \leq \left|\frac{1}{2}\frac{\partial d}{\partial g} - \frac{\partial d}{\partial r}C\right| + \frac{K}{V}.$$

The smallness of the deviation $(J - C)$ would be established if we have proved the smallness of the derivatives $\frac{\partial d}{\partial g}$ and $\frac{\partial d}{\partial r}$. In our work [5] it was shown that

$$|d| \leq \mathcal{E}\left(\frac{1}{V}\right) \to 0 \quad \text{as} \quad V \to \infty.$$

Let us now strengthen this result and estimate the smallness of the derivatives. We notice that

$$\frac{\partial^2 f_\Gamma}{\partial g^2} \leq 0, \qquad \frac{\partial^2 f_\Gamma}{\partial r^2} \leq 0$$

therefore

$$\frac{\partial^2 d}{\partial g^2} \geq \frac{\partial^2 f_{\Gamma_0}}{\partial g^2}, \qquad \frac{\partial^2 d}{\partial r^2} \geq \frac{\partial^2 f_{\Gamma_0}}{\partial r^2}. \tag{17}$$

Let us find expressions for $\frac{\partial^2 f_{\Gamma_0}}{\partial g^2}$ and $\frac{\partial^2 f_{\Gamma_0}}{\partial r^2}$. For this purpose the approximating Hamiltonian Γ_0 can be diagonalized by means of the canonical transformation. Let us introduce new operators $\alpha_f, \alpha_f^\dagger$

$$\alpha_f = a_f u_f + a_{-f}^\dagger v_f, \qquad \alpha_f^\dagger = a_f^\dagger u_f + a_{-f} v_f \tag{18}$$

where

$$u_f = \frac{1}{\sqrt{2}}\sqrt{1+\frac{T_f}{E_f}}, \quad v_f = -\frac{\varepsilon(f)}{\sqrt{2}}\sqrt{1-\frac{T_f}{E_f}}, \quad E_f = \sqrt{(2Cg+r)^2\lambda_f^2 + T_f^2},$$

$$u_f = u_{-f}, \qquad v_{-f} = -v_f.$$

Then we get

$$\Gamma_0 = \sum_{(f)} E_f \alpha_f^\dagger \alpha_f + \frac{1}{2}\sum_{(f)}(T_f - E_f) + 2VC^2 g.$$

The corresponding expression for the free energy per unit volume has the form [3]

$$f_{\Gamma_0} = 2C^2 g - \frac{1}{2V}\sum_{(f)}(E_f - T_f) - \frac{\theta}{V}\sum_{(f)}\ln(1+e^{-\frac{E_f}{\theta}}). \tag{19}$$

Let us write down an equation for C

$$\langle \alpha_f^\dagger \alpha_f \rangle = (e^{\frac{E_f}{\theta}} + 1)^{-1}$$

With the help of the reverse transformation

$$a_f = u_f \alpha_f - v_f \alpha_{-f}^\dagger, \qquad a_f^\dagger = u_f \alpha_f^\dagger - v_f \alpha_{-f}$$

we obtain

$$\langle a_{-f} a_f \rangle_{\Gamma_0} = \langle \alpha_f^\dagger \alpha_f - \alpha_{-f}\alpha_{-f}^\dagger \rangle_{\Gamma_0} u_f v_f = \frac{\lambda_f(2Cg+r)}{2E_f}\tanh\left(\frac{E_f}{2\theta}\right).$$

[3]An alternative approach, in which the volume V was assumed to be infinite from the very beginning in order to avoid the passage to the limit as $V \to \infty$, was developed in [11].

Thus

$$C = \langle J \rangle_{\Gamma_0} = \frac{1}{2V}\sum_{(f)} \lambda_f \langle a_{-f} a_f \rangle_{\Gamma_0}$$

and finally

$$C = \frac{1}{4V}\sum_{(f)} \frac{\lambda_f^2 (2Cg + r)}{E_f} \tanh\left(\frac{E_f}{2\theta}\right) \tag{20}$$

$$Cg + r/2 \geq 0, \quad \text{so} \quad C \geq 0.$$

Let us write down the first and the second derivatives of the free energy with respect to the parameter g

$$\frac{\partial f_{\Gamma_0}}{\partial g} = \frac{1}{V}\langle -2VC(J^\dagger + J) + 2VC^2 \rangle_{\Gamma_0} = -2C^2,$$

$$\frac{\partial^2 f_{\Gamma_0}}{\partial g^2} = -4C\frac{\partial C}{\partial g}.$$

It is convenient to introduce a function

$$\mathcal{F}(C, r, \theta) = \frac{1}{2V}\sum_{(f)} \frac{\lambda_f^4}{2E_f^3}\left\{\frac{\sinh\frac{E_f}{\theta} - \frac{E_f}{\theta}}{\cosh^2\frac{E_f}{2\theta}}\right\} > 0, \tag{21}$$

$$\text{where} \qquad E_f = \sqrt{(2Cg + r)^2 \lambda_f^2 + T_f^2}.$$

After the passage to the limit $V \to \infty$ this function becomes

$$\mathcal{F}(C, r, \theta) = \frac{1}{(2\pi)^3}\int \frac{\lambda^4(p)\left\{\sinh\frac{E_p}{\theta} - \frac{E_p}{\theta}\right\}}{2E_p^3 \cosh^2\frac{E_p}{2\theta}} d\vec{p}.$$

Taking into account Eq.(20) for C and Eq.(21), we obtain

$$\frac{\partial C}{\partial g} = \frac{2C\left(\frac{\partial C}{\partial g} g + C\right)}{2Cg + r} - \mathcal{F}(C, r, \theta)(2Cg + r)^2\left(\frac{\partial C}{\partial g} g + C\right)$$

from where

$$\frac{\partial C}{\partial g} = \frac{\frac{2C^2}{2Cg + r} + \mathcal{F}(C, r, \theta)(2Cg + r)^2 C}{1 - \frac{2Cg}{2Cg + r} + \mathcal{F}(C, r, \theta)(2Cg + r)^2 g}.$$

Thus

$$\frac{\partial^2 f_{\Gamma_0}}{\partial g^2} = 4C\frac{\partial C}{\partial g}$$

and we arrive at the estimation

$$\left|\frac{\partial^2 f_{\Gamma_0}}{\partial g^2}\right| \le \frac{\frac{8C^3}{2Cg+r}}{1-\frac{2Cg}{2Cg+r}} \le \frac{8C^3}{r} \le \frac{8K_1^3}{\delta}, \quad \delta < r \tag{22}$$

Taking into account that

$$|J| \le K_1, \quad K_1 = \text{const}, \quad \langle J\rangle_{\Gamma^0} = C \le K_1,$$

we can find that

$$\left|\frac{\partial^2 f_{\Gamma_0}}{\partial r^2}\right| \le D(\delta), \quad \text{we choose} \quad D(\delta) = \frac{8K_1^3 + 2K_1}{\delta}. \tag{23}$$

Thus, from Eqs.(17, 22, 23) it follows that

$$\frac{\partial^2 d}{\partial g^2} \ge -D(\delta), \quad \frac{\partial^2 d}{\partial r^2} \ge -D(\delta). \tag{24}$$

Lemma

Let $|d(x)| \le \mathcal{E}\left(\frac{1}{V}\right)$, $\mathcal{E}\left(\frac{1}{V}\right) \to 0$ as $V \to \infty$ in the domain $x \ge \delta > 0$ and $\left|\frac{\partial d}{\partial x}\right|$ is a bounded function of x and

$$\frac{\partial^2 d}{\partial x^2} \ge -D(\delta)$$

then the following inequality holds:

$$\left|\frac{\partial d}{\partial x}\right| \le 2\sqrt{2\mathcal{E}\left(\frac{1}{V}\right)D(\delta)}, \quad (x \ge \delta > 0)$$

from which the inequalities result

$$\left|\frac{\partial d}{\partial r}\right| \le 2\sqrt{2\mathcal{E}\left(\frac{1}{V}\right)D(\delta)}, \quad \left|\frac{\partial d}{\partial g}\right| \le 2\sqrt{2\mathcal{E}\left(\frac{1}{V}\right)D(\delta)}.$$

Therefore

$$|\langle(J-C)(J^\dagger-C)\rangle_\Gamma| \le \mathcal{E}_0(1/V,\delta), \quad |\langle(J^\dagger-C)(J-C)\rangle_\Gamma| \le \mathcal{E}_0(1/V,\delta) \tag{25}$$

where $\mathcal{E}_0(1/V,\delta) \to 0$ as $V \to \infty$ for any fixed $\delta > 0$. If one uses one more canonical transformation for the new Fermi operators (18) then the equations of motion (15) takes the form

$$i\frac{d\alpha_f^\dagger}{dt} + \Omega_f\alpha_f^\dagger = R_f, \quad i\frac{d\alpha_f}{dt} - \Omega_f\alpha_f = -R_f^\dagger,$$

$$R_f = 2\lambda_f g\{u_f(J-C)a_{-f} + v_f a_f^\dagger(J^\dagger - C)\}$$

Taking into account inequalities (25), it is possible to show that

$$\begin{aligned} |\langle\alpha_f^\dagger(t)\alpha_f(\tau)\rangle_\Gamma - e^{i\Omega_f(t-\tau)}\langle\alpha_f^\dagger(0)\alpha_f(0)\rangle_\Gamma| &\leq \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}\cdot|t-\tau| \\ |\langle\alpha_f(\tau)\alpha_f^\dagger(t)\rangle_\Gamma - e^{i\Omega_f(t-\tau)}\langle\alpha_f(0)\alpha_f^\dagger(0)\rangle_\Gamma| &\leq \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}\cdot|t-\tau| \end{aligned} \tag{26}$$

and also

$$\begin{aligned} &|\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma - e^{-i\Omega_f(t+\tau)}\langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma| \leq \\ &\leq \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}(|t|+|\tau|) + \mathcal{E}\left(\frac{1}{V},\delta\right)|t|\cdot|\tau|, \end{aligned} \tag{27}$$

where $\mathcal{E}_1\left(\frac{1}{V},\infty\right) \to 0$ as $V \to \infty$ for any fixed $\delta > 0$. From the other side, all correlation averages, in particular $\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma$, depend on t,τ only through the difference $(t-\tau)$. From Eq.(27) we have

$$|\langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma|\cdot|1-e^{-2i\Omega_f t}| \leq 2\sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}|t| + \mathcal{E}_1\left(\frac{1}{V},\delta\right)|t|^2$$

and allowing here an arbitrary parameter t to be equal to $\frac{\pi}{2\Omega_f}$ we get

$$|\langle\alpha_f(0)\alpha_{-f}(0)\rangle_\Gamma| \leq \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}\frac{\pi}{2\Omega_f} + \mathcal{E}_1\left(\frac{1}{V},\delta\right)\frac{\pi^2}{8\Omega_f^2}.$$

Consequently it follows from Eq.(27) that

$$|\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_\Gamma| \leq \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}\left\{|t-\tau| + \frac{\pi}{2\Omega_f} + \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}\frac{\pi^2}{8\Omega_f^2}\right\}.$$

It is easy to see that

$$
\begin{aligned}
|\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_{\Gamma_0}| &= 0, \\
|\langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_{\Gamma} - \langle\alpha_f(t)\alpha_{-f}(\tau)\rangle_{\Gamma_0}| &\leq \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)} \times \qquad (28)\\
\times\left\{|t-\tau| + \frac{\pi}{2\Omega_f} + \sqrt{\mathcal{E}_1\left(\frac{1}{V},\delta\right)}\frac{\pi^2}{8\Omega_f^2}\right\}.
\end{aligned}
$$

Here, under the sign of averaging $\langle ...\rangle$, the time-dependent operators $\alpha_f, \alpha_f^\dagger$ satisfy equations of motion for the Hamiltonians Γ and Γ_0 correspondingly.

Let us return to inequalities (26). Using the spectral representation [15, 16, 17, 18, 19], we can show that

$$
\begin{aligned}
|\langle\alpha_f^\dagger(t)\alpha_f(\tau)\rangle_{\Gamma} - \langle\alpha_f^\dagger(t)\alpha_f(\tau)\rangle_{\Gamma_0}| \leq \mathcal{E}_2\left(\frac{1}{V},\delta\right)|t-\tau| + \mathcal{E}_3\left(\frac{1}{V},\delta\right) \\
\qquad (29)\\
|\langle\alpha_f(\tau)\alpha_f^\dagger(t)\rangle_{\Gamma} - \langle\alpha_f^\dagger(\tau)\alpha_f(t)\rangle_{\Gamma_0}| \leq \mathcal{E}_2\left(\frac{1}{V},\delta\right)|t-\tau| + \mathcal{E}_3\left(\frac{1}{V},\delta\right)
\end{aligned}
$$

where $\mathcal{E}_2\left(\frac{1}{V},\delta\right)$, $\mathcal{E}_3\left(\frac{1}{V},\delta\right)$ tend to zero as $V\to\infty$ for any fixed positive δ. Going back to the original Fermi operators $a_f, a_f^\dagger$ and taking into account Eqs.(26-29), we have

$$
|\langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma} - \langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma_0}| \leq \eta\left(\frac{1}{V},\delta\right)|t-\tau| + \eta'\left(\frac{1}{V},\delta\right), \qquad (30)
$$

where $\mathcal{A},\mathcal{B} = a_f, a_f^\dagger, a_{-f}, a_{-f}^\dagger$ and $\eta\left(\frac{1}{V},\delta\right)\to 0$, $\eta'\left(\frac{1}{V},\delta\right)\to 0$, $V\to\infty$ for any fixed value $\delta > 0$. We must stress that the inequalities (30), as well as all similar inequalities derived before, hold true for $r\geq\delta$. The average $\langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma_0}$ can also be calculated and one can show that

$$
\lim_{r\to 0}\left\{\lim_{V\to\infty}\langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma_0}\right\} = \lim\langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{H_0} \qquad (31)
$$

where $H^0 = \Gamma^0(r=0)$. Therefore, from Eq.(30) follows the existence of the limiting value

$$
\lim_{r\to 0}\left\{\lim_{V\to\infty}\langle\mathcal{A}(t)\mathcal{B}(\tau)\rangle_{\Gamma}\right\} \qquad (32)
$$

which equals to the value on the right hand side of Eq.(31). But it is seen that the average given by Eq.(32) is a quasi-average defined in the sense of work [20]. Hence we can see that the quasi-average (32) is asymptotically equal to the quasi-average for the approximating Hamiltonian H_0. Asymptotically exact expressions for the Green's functions can be derived in full analogy with the expressions for the correlation functions by means of the same technique.

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